Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 10, NUMBER 2 (2007) ISSN 1311-0454

REPRESENTATIONS OF INVERSE FUNCTIONS BY THE INTEGRAL TRANSFORM WITH THE SIGN KERNEL

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Abstract

In this paper we give practical and numerical representations of inverse functions by using the integral transform with the sign kernel, and show corresponding numerical experiments by using computers. We derive a very simple formula from a general idea for the representation of the inverse functions, based on the theory of reproducing kernels.

Mathematics Subject Classification: Primary 30C40

Key Words and Phrases: inverse function, reproducing kernel, Sobolev space, sign function, integral transform, solution method of general equations

1. Introduction

Considering arbitrary mapping ϕ from an arbitrary abstract set into another arbitrary set, the third author of this paper tried to represent the inversion ϕ^{-1} in terms of the direct mapping ϕ, and following the general ideas from [1], surprisingly obtained some simple concrete inverse formulas.

This paper establishes an analytical and direct relation between a monotone function and its inversion. By the invariance relation principle for analytic functions, we could derive new local and analytical representations of the inverse functions.

The method and ideas proposed for the inversion formulas concern rather general non-linear mappings, however, their formulations sound quite involved for such generality. So, we shall state the results concretely for typical cases of mappings and shall give some corresponding numerical experiments. First, we discuss the principle of our new method for the representations of inverses of non-linear mappings, based on the ideas from [1]:

We shall consider some representation of the inversion ϕ^{-1} in integral form, at this moment needing a natural assumption for the mapping ϕ. Then, we shall transform the integral representation by the mapping ϕ to the original space, that is the defined domain of the mapping ϕ. Then, we will be able to obtain the representation of the inverse ϕ^{-1} in terms of the direct mapping ϕ. In [1], we considered the representation of the inverse ϕ^{-1} in some reproducing kernel Hilbert spaces. However, here we shall consider the representations of the inverse ϕ^{-1} for a very concrete situation and we shall give a very fundamental representation of the inverse for some general functions on 1-dimensional spaces. Actually, in this paper, we shall consider the problems by using a simple Sobolev and reproducing kernel space, and in the next section we shall remind the basic background. By using the representation of the functions in the reproducing kernel Hilbert space, we will be able to obtain some natural representation formulas of the inverses of some general and reasonable functions.

2. Background results

Note that

$$K(y_1, y_2) = \frac{1}{2} e^{-|y_1 - y_2|}, \qquad y_1, \quad y_2 \in [A, B] \tag{2.1}$$

is the reproducing kernel in the Sobolev-Hilbert space H_K whose members are real-valued and absolutely continuous functions on $[A, B]$ and whose inner product is given by (see [2]):

$$(f_1, f_2)_{H_K} = \int_A^B \big(f_1'(y) f_2'(y) + f_1(y) f_2(y)\big) dy + f_1(A) f_2(A) + f_1(B) f_2(B). \tag{2.2}$$

3. Integral representations of inverse functions

We start with the most typical case.

Consider a function $y = f(x)$ of the C^1-class, strictly increasing, and with $f'(x)$ not vanishing on $[a, b]$ $(f(a) = A, f(b) = B)$. Then, the inverse function $f^{-1}(y)$ is a single-valued function and belongs to the space H_K, and from the reproducing property we obtain, for any $y_0 \in [f(a), f(b)]$ the representation:

$$f^{-1}(y_0) = \big(f^{-1}(\cdot), K(\cdot, y_0)\big)_{H_K}$$

$$= \int_{f(a)}^{f(b)} \big((f^{-1})'(y)K_y(y, y_0) + f^{-1}(y)K(y, y_0)\big)dy + aK(f(a), y_0) + bK(f(b), y_0). \tag{3.1}$$

Note the identity

$$K(y_1, y_2) = \frac{1}{2}e^{-|y_1-y_2|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi(y_1-y_2)}}{\xi^2+1}d\xi,$$

and so, from $f(a) \le y_0 \le f(b)$, we obtain

$$f^{-1}(y_0) = \frac{1}{2\pi}\int_{f(a)}^{f(b)}\left((f^{-1})'(y)\int_{-\infty}^{\infty} i\xi\frac{e^{i\xi(y-y_0)}}{\xi^2+1}d\xi + f^{-1}(y)\int_{-\infty}^{\infty}\frac{e^{i\xi(y-y_0)}}{\xi^2+1}d\xi\right)dy$$

$$+\frac{a}{2}e^{(f(a)-y_0)} + \frac{b}{2}e^{-(f(b)-y_0)}. \tag{3.2}$$

The above integral is calculated by the transformation $y = f(x)$ of the variables y, x as follows:

$$\frac{1}{2\pi}\int_a^b\left(\int_{-\infty}^{\infty} i\xi\frac{e^{i\xi(f(x)-y_0)}}{\xi^2+1}d\xi + xf'(x)\int_{-\infty}^{\infty}\frac{e^{i\xi(f(x)-y_0)}}{\xi^2+1}d\xi\right)dx \tag{3.3}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_a^b\left(i\xi\frac{e^{i\xi(f(x)-y_0)}}{\xi^2+1} + xf'(x)\frac{e^{i\xi(f(x)-y_0)}}{\xi^2+1}\right)dx\right)d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\xi^2+1}\int_a^b\left(i\xi e^{i\xi(f(x)-y_0)} + xf'(x)e^{i\xi(f(x)-y_0)}\right)dxd\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\xi^2+1}\left(i\xi\int_a^b e^{i\xi(f(x)-y_0)}dx - \frac{i}{\xi}\Big[xe^{i\xi(f(x)-y_0)}\Big]_a^b\right.$$

$$\left. + \frac{i}{\xi}\int_a^b e^{i\xi(f(x)-y_0)}dx\right)d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\xi^2+1}\left(i\left(\xi+\frac{1}{\xi}\right)\int_a^b e^{i\xi(f(x)-y_0)}dx - \frac{i}{\xi}\left(be^{i\xi(f(b)-y_0)} - ae^{i\xi(f(a)-y_0)}\right)\right)d\xi$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\xi}\int_a^b e^{i\xi(f(x)-y_0)}dxd\xi + \frac{ia}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi(f(a)-y_0)}}{\xi(\xi^2+1)}d\xi - \frac{ib}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi(f(b)-y_0)}}{\xi(\xi^2+1)}d\xi$$

$$= \frac{i}{2\pi}\int_a^b \int_{-\infty}^{\infty} \frac{e^{i\xi(f(x)-y_0)}}{\xi} d\xi dx + \frac{ia}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi(f(a)-y_0)}}{\xi(\xi^2+1)} d\xi - \frac{ib}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi(f(b)-y_0)}}{\xi(\xi^2+1)} d\xi$$

$$= -\frac{1}{2\pi}\pi\int_a^b \operatorname{sign}(f(x)-y_0)dx - \frac{a}{2\pi}2\pi e^{-\frac{1}{2}(f(a)-y_0)\operatorname{sign}(f(a)-y_0)}\sinh(\frac{f(a)-y_0}{2})$$

$$+\frac{b}{2\pi}2\pi e^{-\frac{1}{2}(f(b)-y_0)\operatorname{sign}(f(b)-y_0)}\sinh(\frac{f(b)-y_0}{2})$$

$$= -\frac{1}{2}\int_a^b \operatorname{sign}(f(x)-y_0)dx - \frac{a}{2}\big(e^{\frac{f(a)-y_0}{2}} - e^{-\frac{f(a)-y_0}{2}}\big)e^{-\frac{1}{2}(f(a)-y_0)\operatorname{sign}(f(a)-y_0)}$$

$$+\frac{b}{2}\big(e^{\frac{f(b)-y_0}{2}} - e^{-\frac{f(b)-y_0}{2}}\big)e^{-\frac{1}{2}(f(b)-y_0)\operatorname{sign}(f(b)-y_0)}$$

$$= \frac{1}{2}\int_a^b \operatorname{sign}(y_0 - f(x))dx - \frac{a}{2}\big(e^{\frac{f(a)-y_0}{2}} - e^{-\frac{f(a)-y_0}{2}}\big)e^{\frac{1}{2}(f(a)-y_0)}$$

$$+\frac{b}{2}\big(e^{\frac{f(b)-y_0}{2}} - e^{-\frac{f(b)-y_0}{2}}\big)e^{-\frac{1}{2}(f(b)-y_0)}$$

$$= \frac{1}{2}\int_a^b \operatorname{sign}(y_0 - f(x))dx - \frac{a}{2}\big(e^{(f(a)-y_0)} - 1\big) + \frac{b}{2}\big(1 - e^{-(f(b)-y_0)}\big)$$

$$= \frac{a+b}{2} + \frac{1}{2}\int_a^b \operatorname{sign}(y_0 - f(x))dx - \frac{a}{2}e^{(f(a)-y_0)} - \frac{b}{2}e^{-(f(b)-y_0)}.$$

The above singular integrals are considered in the sense of the Cauchy principal values.

Thus, we obtain the representation

$$f^{-1}(y_0) = \frac{a+b}{2} + \frac{1}{2}\int_a^b \operatorname{sign}\big(y_0 - f(x)\big)dx. \tag{3.4}$$

Using reproducing kernel Hilbert spaces from [2] as in (3.1), we calculate similarly with the related assumptions, but surprisingly enough, we obtain the same formula (3.4). For this formula (3.4), we note that we do not need any smoothness assumptions for the function $f(x)$. Indeed, we need only the strictly increasing assumption. Even the assumption of integrability is not needed.

4. Numerical experiments

We give here several numerical examples. The integrals have been processed easily by *Mathematica*.

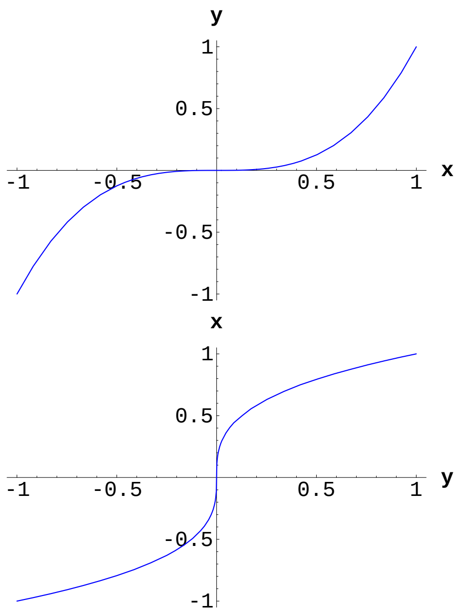


Figure 1. The graph of inverse function for $y = f(x) = x^3$.

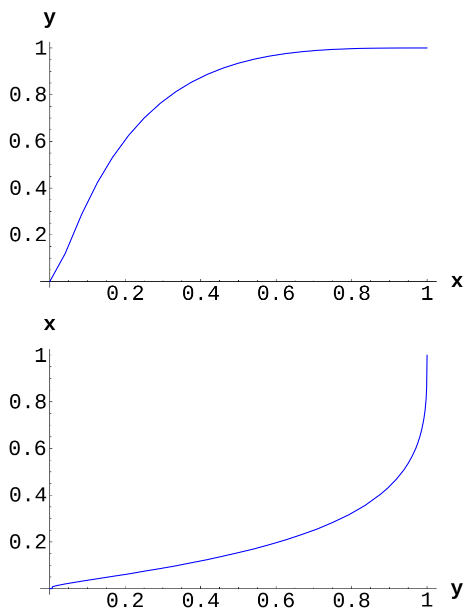


Figure 2. The graph of inverse function for

$$y = f(x) = \begin{cases} x(2 - \sin\log x - \cos\log x) & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$$

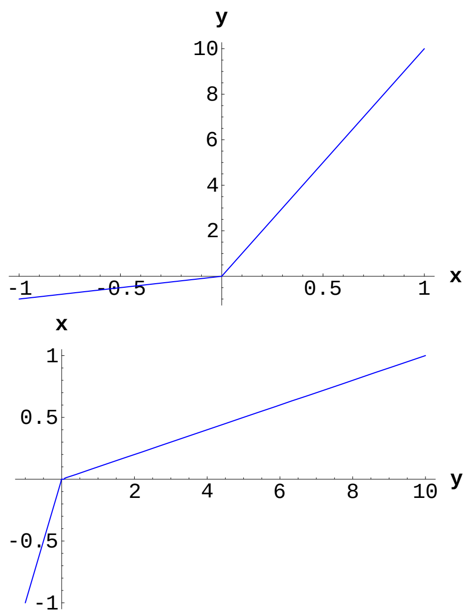
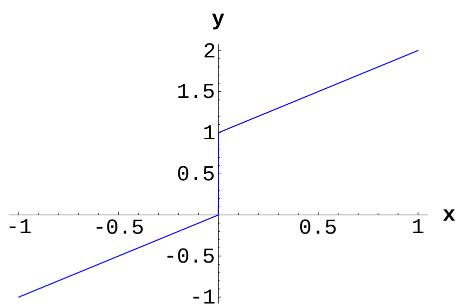


Figure 3. The graph of inverse function for

$$y = f(x) = \begin{cases} x & x \le 0 \\ 10x & x > 0 \end{cases}$$



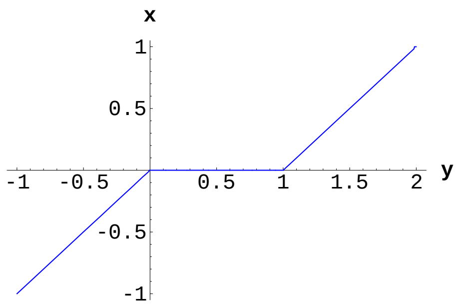


Figure 4. The graph of inverse function for

$$y = f(x) = \begin{cases} x & x \le 0 \\ x+1 & x > 0 \end{cases}$$

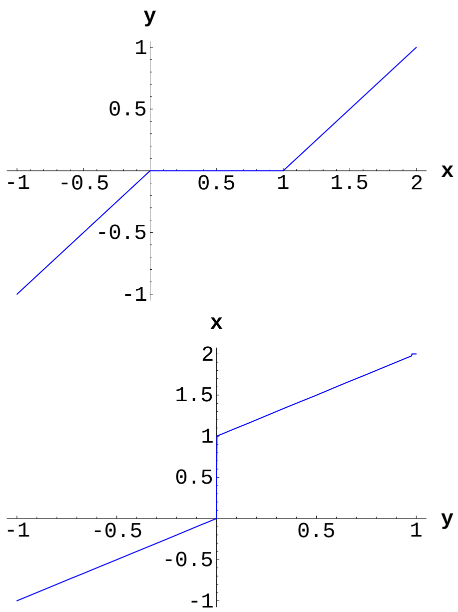


Figure 5. The graph of inverse function for

$$y = f(x) = \begin{cases} x & x \le 0 \\ 0 & 0 < x \le 1 \\ x-1 & x > 1 \end{cases}$$

Acknowledgements: The second author, T. Matsuura is supported in part by the Grant-in-Aid for Scientific Research (C)(No.18540110) from the Japan Society for the Promotion Science. The third author, S. Saitoh is supported in part by the Grant-in-Aid for Scientific Research (C)(2)(No. 19540164) from the Japan Society for the Promotion Science.

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Received: September 9, 2007

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